Recursively defining base b expansion calculators

Our CSE 20 professor defined a bunch of procedures in pseudocode using rather imperative constructs such as `while` loops, even though it could be defined using some epic iterative recursion instead.

```
procedure half(n : a positive integer)
r := 0
while n > 1
    r := r + 1
    n := n − 2
return r {r holds the result of the operation}
```

This could instead be defined with a piecewise function.

$$\mathrm{half}' : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$
$$\mathrm{half}'(r, n) = \begin{cases} \mathrm{half}'(r+1, n-2) & \text{if } n > 1 \\ r & \text{otherwise} \end{cases}$$

$$\mathrm{half} : \mathbb{N} \to \mathbb{N}$$
$$\mathrm{half}(n) = \mathrm{half}'(0, n)$$

I'm going to generalize half to any divisor. It'll be the equivalent to the **div** operator, which does integer division.

$$\mathrm{div}' : \mathbb{N} \times \mathbb{N} \times \mathbb{Z}^+ \to \mathbb{N}$$
$$\mathrm{div}'(q, n, d) = \begin{cases} \mathrm{div}'(q+1, n-d, d) & \text{if } n \geq d \\ q & \text{otherwise} \end{cases}$$

$$\mathrm{div} : \mathbb{N} \times \mathbb{Z}^+ \to \mathbb{N}$$
$$\mathrm{div}(n, d) = \mathrm{div}'(0, n, d)$$

Because $n = dq + r = d(n \ \mathbf{div}\ d) + (n \ \mathbf{mod}\ d)$, let's define a mod function.

$$\mathrm{mod} : \mathbb{N} \times \mathbb{Z}^+ \to \mathbb{N}$$
$$\mathrm{mod}(n, d) = n - d \cdot \mathrm{div}(n, d)$$

That's pretty poggers. My professor defined a procedure that calculates the integer part of $\log_b$. As another epic piecewise function,

$$B = \left\{b \in \mathbb{Z}^+ \,\middle|\, b > 1\right\}$$

$$\log' : \mathbb{N} \times \mathbb{N} \times B \to \mathbb{N}$$
$$\log'(r, n, b) = \begin{cases} \log'(r + 1, \operatorname{div}(n, b), b) & \text{if } n > b - 1 \\ r & \text{otherwise} \end{cases}$$

$$\log : \mathbb{N} \times B \to \mathbb{N}$$
$$\log(n, b) = \log'(0, n, b)$$

Before I can convert numbers to bases, I'll first define the set of all base b expansions of a natural number.

First, let $D_b = \{d \in \mathbb{N} \,|\, d < b\}$, the set of digits available. $D_b \subset E_b$. Then if $e \in E_b$, $d \in D_b$, and $e \neq 0$, then $e \circ d \in E_b$. Also, for convenience, let $E_b \subset E_b'$ and $\lambda \in E_b'$; in other words, E_b' is E_b but with the empty string.

Finally, I think I'll implement the "Least significant first" algorithm for calculating the base b expansion from the right.

$$\text{base}' : E_b' \times \mathbb{N} \times B \to E_b$$
$$\text{base}'(a, q, b) = \begin{cases} \text{base}'(\operatorname{mod}(q, b) \circ a, \operatorname{div}(q, b), b) & \text{if } q \neq 0 \\ a & \text{otherwise} \end{cases}$$

$$\text{base} : \mathbb{N} \times B \to E_b$$
$$\text{base}(n, b) = \begin{cases} 0 & \text{if } n = 0 \\ \text{base}'(\lambda, n, b) & \text{otherwise} \end{cases}$$

…I think? CSE 20 is fun.

□