good morning. this is an equation sheet for ece 45 based on past lectures and quizzes. as it turns out, it is not efficient to have notes and my quiz work in the same notebook, nor is it efficient to keep copying formulas onto every page

also here's some dirac delta properties because I keep mixing them up:

$$x(t)\delta(t-t_0) = x(t_0)\delta(t) \text{ (dirac delta w/ area } x(t_0)\text{)}$$

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

$$x(t) * \delta(t-a) = x(t-a)$$

one

zeger really finds these quite delicious. i will call these zeger fetishes

$$2\cos t = e^{jt} + e^{-jt}$$

$$2j\sin t = e^{jt} - e^{-jt}$$

two

to prove **linear**, is passing $Ax_1(t) + Bx_2(t)$ through the system the same as passing x_1 and x_2 individually and then doing $Ay_1(t) + By_2(t)$ on their outputs?

to prove **time-invariant**, is passing $\hat{x}(t-t_0)$ the same as passing $\hat{x}$ and then

tip remove the extra coefficients first in case the magic happens far outside the Desmos view window

three

fourier series.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

where $\omega_0 = \frac{2\pi}{T}$

fourier coefficients. can be complex

$$F_n = \frac{1}{T}\int_T f(t)e^{-jk\omega_0 t}dt$$

for sinusoids, you should use zeger fetishes to turn them into e^{jt}'s, which fit nicely with the $n\omega_0$'s in the fourier series thingy.

trig form $\frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi nt)}{n}$

expt form $\frac{1}{2} + \frac{j}{2\pi}\sum_{n=1}^{\infty}\frac{e^{j2\pi nt}}{n} + \frac{j}{2\pi}\sum_{n=-1}^{-\infty}\frac{e^{j2\pi nt}}{n}$

time-shift property	$f(t-t_0) \leftrightarrow F_n e^{jn\omega_0 t_0}$
derivative property	$f'(t) \leftrightarrow (jn\omega_0)F_n$
multiplication property	$f(t)g(t) \leftrightarrow \sum_{k=-\infty}^{\infty} F_k G_{n-k}$ (discrete convolution sum)
parseval's theorem	$\frac{1}{T}\int_T \lvert f(t)\rvert^2 dt \leftrightarrow \sum_{n=-\infty}^{\infty} \lvert F_n\rvert^2$
$\rightarrow$	$f^*(t) \leftrightarrow F^*_{-n}$

$$X_n \rightarrow \boxed{H(\omega)} \rightarrow X_n H(n\omega_0)$$

don't forget that for $\sin/\cos$, most of the coefficients are 0, so can just deal with them manually
also if you're getting a zero where you shouldn't, you probably made a sign error. redo it
and don't forget that to find the magnitude of a complex number, square the components, not j. ie you shouldn't be doing j^2. fool

example fourier series

here are some EXAMPLES because screw derivation, using resources $>>>$

$f(t)$	F_n
$\cos(kt)$	$F_{\pm 1} = \frac{1}{2}$, others 0
$\sin(kt)$	$F_{-1} = \frac{1}{2j}$, $F_1 = -\frac{1}{2j}$, others 0
$\lvert\sin t\rvert$	$\frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos(2nt)}{4n^2-1} = \frac{2}{\pi}\sum_{n=-\infty}^{\infty} \frac{e^{2jnt}}{1-4n^2}$
triangle wave*	$F_0 = \frac{1}{2}$, others $\frac{j}{2\pi n}$

*triangle wave is $f(t) = t$ between 0 and 1, and it repeats

four

fourier transform of $f(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

inverse fourier transform of $F(\omega)$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$$X(\omega) \rightarrow \boxed{H(\omega)} \rightarrow X(\omega)H(\omega)$$

$$\operatorname{sinc} t = \frac{\sin t}{t} \text{ (and } \operatorname{sinc} 0 = 1\text{)}$$

rect is a unit square (so it's 1 between $-\frac{1}{2}$ and $\frac{1}{2}$)
if

$$\delta(t) \to \boxed{H(\omega)} \to h(t)$$

then

$$x(t) \to \boxed{H(\omega)} \to \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \text{ (convolution integrals)}$$

$$\int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt = x(t_0)$$

spencer covered this. $x(t)\delta(t-t_0)$ is a dirac delta with area $x(t_0)$
zeger says $u(0)$ can be 0 or 1 but he initially defined it to be 0 (and then graphed it at 1??)

$$\begin{aligned}
\mathrm{rect}(\frac{t}{t_0}) &\leftrightarrow t_0 \,\mathrm{sinc}(\frac{\omega t_0}{2}) \\
\delta(t) &\leftrightarrow 1 \\
1 &\leftrightarrow 2\pi\delta(t) \\
f(t-t_0) &\leftrightarrow F(\omega)e^{-j\omega t_0} \text{ (SHIFT TIME/FREQUENCY!!!!!!)} \\
f(t)e^{j\omega_0 t} &\leftrightarrow F(\omega-\omega_0)
\end{aligned}$$

camel recommends using a table (THIS IS A LINK) for these

five

duality/symmetry property

$$F(t) \leftrightarrow 2\pi f(-\omega)$$

time derivative

$$\begin{aligned}
\frac{df(t)}{dt} &\leftrightarrow j\omega F(\omega) \\
-jtf(t) &\leftrightarrow \frac{dF(\omega)}{d\omega} \\
tf(t) &\leftrightarrow j \cdot \frac{dF(\omega)}{d\omega}
\end{aligned}$$

convolution

$$\begin{aligned}
x(t) * y(t) &= \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau \\
&= \int_{-\infty}^{\infty} x(t-\tau)y(\tau)d\tau
\end{aligned}$$

$$f(t) * \underbrace{g(t)}_{\text{impulse response}} \leftrightarrow F(\omega)G(\omega)$$

$$f(t)g(t) \leftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega)$$

$X^*(t) \leftrightarrow X^*(-\omega)$ signals don't have to be real

$X(-\omega) = X^*(\omega)$ ONLY if $x(t)$ real!!

$f(-t) \leftrightarrow F(-\omega)$ **time reversal**

$x(t)$ real, even $\leftrightarrow X(\omega)$ real, even

$x(t)$ real, odd $\leftrightarrow X(\omega)$ purely imaginary (i.e. $Re[X(\omega)] = 0$), odd

$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$ **parseval's theorem for fourier transforms**

$f(at) \leftrightarrow \frac{1}{|a|}F(\frac{\omega}{a})$ **time scaling** (squishing function $\rightarrow$ higher frequency)

parseval's theorem for fourier transforms

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

some more examples:

$$\cos(\omega_0 t) \leftrightarrow \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$\sin(\omega_0 t) \leftrightarrow \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

$$\text{sinc}(\omega_0 t) \leftrightarrow \frac{\pi}{\omega_0}\,\text{rect}\left(\frac{\omega}{2\omega_0}\right)$$

$$e^{-at}u(t) \leftrightarrow \frac{1}{a + j\omega} \quad \text{(for } a > 0\text{)}$$

$$\frac{1}{a + jt} \leftrightarrow 2\pi e^{a\omega}u(-\omega)$$

some things from god spencer:

— if they pass $\delta(t)$ into system, they're giving you $h(t)$! i.e. the entire system

$$\delta(t) \rightarrow \boxed{H(\omega)} \rightarrow h(t)$$

so $h(t) \leftrightarrow H(\omega)$ and $y(t) = x(t) * h(t)$ (a convolution) $\leftrightarrow X(\omega)H(\omega)$

— "diagram" refers to the $x(t) \rightarrow \boxed{H(\omega)} \rightarrow y(t)$ things

— when multiplying rect funcs, take their intersection

$$\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y))$$

$$x(t) * \delta(t - a) = x(t - a)$$

$(Ax(t) + By(t)) * z(t) = Ax(t) * z(t) + By(t) * z(t)$ linearity of convlution

tip don't forget that cosine is even $\cos t = \cos(-t)$ and sine is odd $-\sin t = \sin(-t)$!!!

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$$e^{j\omega t} \to \boxed{h(t)} \to e^{j\omega t} H(\omega)$$

convolution is commutative, distributive, associative

shift property—$f(t-t_1) * h(t-t_2) = y(t-t_1-t_2)$

derivative property—$y'(t) = f'(t) * h(t) = f(t) * h'(t)$, so $y''(t) = f'(t) * h'(t)$

because commutative, order doesn't matter: $Y(\omega) = X(\omega)G(\omega)H(\omega)$

$$x(t) \to \boxed{g(t)} \to \boxed{h(t)} \to y(t) = x(t) * g(t) * h(t)$$

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

$\delta(t)$ acts as "identity" element

$$f(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}F(\omega-\omega_0) + F(\omega+\omega_0)$$

convolution examples

convolving two squares (side 1, lower left origin) $f(t) = h(t) = \operatorname{rect}\left(t - \frac{1}{2}\right)$ produces a triangle (base 2, height 1, lower left origin)

$$\operatorname{rect}\left(t-\frac{1}{2}\right) * \operatorname{rect}\left(t-\frac{1}{2}\right) = \begin{cases} t & \text{if } 0 < t < 1 \\ 2-t & \text{if } 1 < t < 2 \\ 0 & \text{else} \end{cases}$$

convolving $x(t) = e^{-at}u(t)$, $h(t) = e^{-bt}u(t)$ (downwards exponentials only for positive t) where a, $b > 0$ makes

$$y(t) = \begin{cases} \frac{e^{-at}-e^{-bt}}{b-a} \cdot u(t) & \text{if } a \neq b \\ te^{-bt}u(t) & \text{if } a = b \end{cases}$$

convolving $f(t) = A\operatorname{rect}\left(\frac{t}{2t_0}\right)$ (rectangle of height A centred around origin from $-t_0$ to t_0) with itself produces triangle $g(t)$ centered around origin from $-2t_0$ to $2t_0$ w/ height $2A^2t_0$

$$g(t) \leftrightarrow G(\omega) = F^2(\omega) = 4A^2t_0^2\operatorname{sinc}^2(\omega t_0)$$

fourier transform of fourier series

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \leftrightarrow F(\omega) = \sum_{n=-\infty}^{\infty} F_n \cdot 2\pi\delta(\omega - n\underbrace{\omega_0}_{\frac{2\pi}{T}})$$

sum of deltas w/ coeffs $2\pi F_n$, "discrete"

fourier transform of impulse $s(t)$ (inf sum of equally spaced deltas w/ equal area, maybe starting at 0):

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = S(\omega) = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\underbrace{\omega_0}_{\frac{2\pi}{T}})$$

from god spencer: to convolve, take one func (the "simpler" one, eg with more constants), flip it, then slide it along other func. @ every pt where smth changes, multiply the functions

convolving $h(t)$, a triangle that's $h(t) = t$ only for $0 < t < 1$ and 0 elsewhere, with $u(t)$ produces

$$y(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 < t < 1 \\ \frac{1}{2} & t > 1 \end{cases}$$

and if a different $h_2(t)$ can be expressed in terms of an $h(t)$ we know, then can use convolution properties (above) to avoid doing integral bleh again

don't forget about $Y(\omega) = X(\omega)H(\omega)$, and if finding a specific $y(t_0)$ then can just do $y(t_0) = \int_{-\infty}^{\infty} x(t_0 - \tau)h(\tau)d\tau$ directly (this is actually useful)

seven

$$S(\omega) = \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

where T_s sampling period, $\omega_s = \frac{2\pi}{T_s}$ sampling frequency

if you sample $x(t)$ at integer multiples of period T_s it produces $y(t) = x(t)s(t)$, whose fourier transform is

$$Y(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s) \text{ (FOURIER TRANSFORM OF } x(t)s(t))$$

there are also block diagrams but i don't know how to draw that in latex

$$\begin{array}{c} x(t) \rightarrow \bigotimes \rightarrow \boxed{h(t)} \rightarrow z(t) \\ \uparrow \\ s(t) \end{array}$$

omega	what it means
ω_s	sampling frequency
ω_c	cutoff frequencies of low-pass filter
ω_m	maximum frequency for bandlimited thingy

so the reconstruction filter $H(\omega)$ (an ideal LPF) would be a rect from $-\omega_c$ to ω_c

tip from discussion: if a func is periodic, use its fourier coefficients

for fourier coefficients, $Y_n = X_n H(2\pi n)$ (plug into fourier series formula)

ahhhh

careful! $y(t)/Y(\omega)$ sometimes means $x(t)s(t)$, sometimes means output of the lti system

eight

really hope there's no am radio stuff on the quiz . . .

when finding fourier transform of a func being multiplied by a cos/sin, that's okay. it turns into deltas which are nice

nine

la place transforms!
conventional names: $z = \underbrace{\sigma}_{\text{real}} + j \underbrace{\omega}_{\text{imag component}} \in \mathbb{C}$

for determining whether an exponential explodes or converges,

$$\lim_{t\to\infty} e^{zt} = \begin{cases} 0 & \text{if } \sigma < 0 \\ \infty & \text{if } \sigma > 0 \\ \text{undefined} & \text{if } \sigma = 0 \end{cases}$$

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$X(j\omega)$ is fourier transf (imaginary axis of s-plane)
these are FOURIER transforms:

$$e^{-t}u(t) \leftrightarrow \frac{1}{1+j\omega}$$
$$e^{t}u(t) \leftrightarrow \text{nothing}$$

but these are the tasty LAPLACE transforms:

$$\begin{aligned}
e^{-at}u(t) &\leftrightarrow \frac{1}{s+a} && \text{ROC: } \mathrm{Re}(s) > \mathrm{Re}(-a) && a \in \mathbb{C} \\
-e^{-at}u(-t) &\leftrightarrow \frac{1}{s+a} && \text{ROC: } \mathrm{Re}(s) < \mathrm{Re}(-a) \\
u(t) &\leftrightarrow \frac{1}{s} && \text{ROC: } \mathrm{Re}(s) > 0 \\
-u(-t) &\leftrightarrow \frac{1}{s} && \text{ROC: } \mathrm{Re}(s) < 0 \\
\sin(at)u(t) &\leftrightarrow \frac{a}{s^2+a^2} && \text{ROC: } \mathrm{Re}(s) > 0 && a \in \mathbb{R} \\
\cos(at)u(t) &\leftrightarrow \frac{s}{s^2+a^2} && \text{ROC: } \mathrm{Re}(s) > 0 \\
\delta(t) &\leftrightarrow 1 && \text{ROC: all of } \mathbb{C} \\
e^{-a|t|} &\leftrightarrow \frac{-2a}{s^2-a^2} && \text{ROC: } -\mathrm{Re}(a) < \mathrm{Re}(s) < \mathrm{Re}(a) && a \in \mathbb{C}, \mathrm{Re}(a) > 0
\end{aligned}$$

ROC only cares about real component (so it's composed of vertical lines), and it doesn't include poles
if ROC goes to $-\infty$, anti-causal/left-sided; to ∞, causal/right-sided; otherwise, if bounded between poles, 2-sided

properties

linearity	$ax(t) + by(t) \leftrightarrow aX(s) + bY(s)$ (coefficients complex)
derivative	$-tx(t) \leftrightarrow \frac{dX(s)}{ds}$ (same ROC)
	$t^n x(t) \leftrightarrow (-1)^n \underbrace{X^{(n)}}_{n\text{th derivative}}(s)$
shift in frequency	$e^{at}x(t) \leftrightarrow X(s-a)$
shift in time	$x(t-a) \leftrightarrow e^{-as}X(s)$

partial fractions (for inverse LTs): to find A, multiply both sides by denominator, plug in s to make others zero

$$\frac{1}{(s+2)(s-1)} = \frac{A}{s+2} + \frac{B}{s-1} \rightarrow \frac{s+2}{(s+2)(s-1)} = \frac{A(s+2)}{s+2} + \frac{B(s+2)}{s-1}, s = -2$$

if factor is squared, differentiate then plug in s

tip: use derivative property to turn squares back into fractions, which can use for inverse LTs

$$\underbrace{te^{-a}}_{\text{causal}}, \underbrace{-te^{-at}u(-t)}_{\text{anticausal}} \leftrightarrow \frac{1}{(s+a)^2}$$

tip: if know σ, multiply func by $e^{-\sigma t}$ and see if it blows up as $t \to -\infty$ or ∞ (if so, doesn't converge)